Utility Optimization based on MSE for Parallel Broadcast Channels: The Square Root Law

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***Abstract*— To design efficient online resource allocation algorithms convexity of the optimization problem is an important measure. This paper covers the sum-power constraint utility maximization of parallel broadcast channels and the minimization of the sum-power for minimum utility requirements. We derive a class of utility functions that results in convex formulations of both optimization problems, thereby extending known utility classes. By introducing the square-root criteria we present a straight forward test to check if arbitrary utility functions belong to our class. Besides the utility framework the paper gives some insights on the general structure of the MSE region. Along the way we disprove a result on the general MSE region correcting a former approach.**

I. Introduction

The increasing demand for bandwidth in wireless applications and shortage of spectrum represent fundamental challenges for todays system designers. To allow operators competitive offering of a variety of services an efficient exploration of resources is fundamental. System designers have to design resource allocation rules that exploit resources best possible related to a whole set of optimization metrics. However, this has to be done always under the consideration of the real time ability and complexity of their algorithms. In order to allow a generalized description and variable definition of optimal operation points in wireless communications system the concept of utility optimization seems a promising candidate. It allows combining and weighting of different performance measures like e.g. the system throughput and fairness. In connection with resource constraints like power and bandwidth often a single algorithm is sufficient to solve an overarching family of optimization problems. In case the optimization problem is convex a multitude of ready-to-use algorithms exists for which convergence to the global optimum can be guaranteed in polynomial time. Most times, however, the difficulty is to determine weather an optimization problem is convex or can be transformed into a convex representation, respectively. Although pure throughput maximization for interference limited wireless systems results in general in non-convex optimization problems, combining it with fairness in a utility framework can often transform it into a convex one. For non-orthogonal, interference-limited MIMO systems an important class of utility-functions, that allows a convex problem formulation, was presented in [1]. There, based on the Perron-Frobenius theory, it is shown that for any utility function in dependence of the SINR, where the inverse is log-convex, a convex problem formulation can be found. In this paper we derive a new class of utility functions based on the Mean Square Error (MSE) that also results in a convex problem formulation. It is adapted to parallel broadcast (BC) channels such as multiuser OFDM with a sum power constraint. Our class extends the log-convex set in the direction of more throughput oriented metrics. This is especially interesting since it includes allocations where users are not assigned any resources at the optimum, which cannot happen with utilities from the log-convex class.

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II. System Model

Consider a system of K parallel broadcast channels over which a transmitter communicates with M receivers. More general, any set of K parallel Gaussian BC can be assumed. We denote the set of subchannels and users by $\mathcal{K}$ and $\mathcal{M}$, respectively. No assumptions are made on the number of users to be allocated to each of the K subchannels. The transmitter is restricted to linear signal processing. Then the system can be represented by

$$y_{m,k} = h_{m,k} \sum_{n \in \mathcal{M}} x_{n,k} + n_{m,k} \quad \forall\, m \in \mathcal{M}, k \in \mathcal{K} \tag{1}$$

where $h_{m,k}$ is the channel between the transmitter and user m on subchannel k, $x_{m,k}$ and $y_{m,k}$ are the signals transmitted to and received by user m, respectively. The received signals are corrupted by circular symmetric white Gaussian noise $n_{m,k} \sim \mathcal{CN}(0,1)$. Due to technical limitations the system is limited by a sum power constraint. Therefore, for the expectation value of the power of the symbols holds

$$\sum_{k \in \mathcal{K}, m \in \mathcal{M}} \mathrm{E}\{|x_{m,k}|^2\} \leq \bar{P}.$$

With $g_{m,k} = |h_{m,k}|^2$ the achievable signal to interference plus noise-ratio (SINR) on subchannel k is given by

$$\beta_{m,k} = \frac{g_{m,k} p_{m,k}}{g_{m,k} \sum_{n \neq m} p_{n,k} + 1},$$

where $p_{m,k}$ is the power allocated to user m on subchannel k. With the normalized MSE [2], [3]

$$\mathrm{MSE}_{m,k} = 1 - \frac{g_{m,k} p_{m,k}}{g_{m,k} \sum_{n\in\mathcal{M}} p_{n,k} + 1}, \tag{2}$$

we define for some technical reason the complementary MSE of user m on subchannel k as

$$\begin{aligned}\gamma_{m,k} &= 1 - \mathrm{MSE}_{m,k} \\ &= \frac{g_{m,k} p_{m,k}}{g_{m,k} \sum_{n\in\mathcal{M}} p_{n,k} + 1}.\end{aligned} \tag{3}$$

Hence, the minimization problem of MSEs turns into maximization of complementary MSEs. Under the assumption that the optimum linear (MMSE) receive filter is applied, the following well-known bijective mapping

$$\gamma_{m,k} = \frac{\beta_{m,k}}{1+\beta_{m,k}} \tag{4}$$

relates both performance measures to one another. Equivalently the SINR can be represented in terms of the complementary MSE by

$$\beta_{m,k} = \frac{\gamma_{m,k}}{1-\gamma_{m,k}}.$$

Now, in analogy to the well-established sum-MSE, define the sum of the complementary MSEs as

$$\gamma_m = \sum_{k\in\mathcal{K}} \gamma_{m,k} \quad m \in \mathcal{M}\,. \tag{5}$$

Note that the normalization of the MSEs on each subchannel yields the upper bound by $\gamma_m \le K$ for all m.

Remark 1: Due to recent duality results (see Sec. IV) all our derivations (even though more difficult to see) can be equivalently formulated for a multiple access channel (MAC), hence, extending the possible range of applications for our results. However, due to the appealing structure of parallel BC channels with single antennas we stick to the BC channel in this paper.

III. Problem Statement

Users' QoS demands can be described by some appropriate utility functions that map the used resources into a real number. Here, we are interested in utility maximization (Problem 1) or sum power minimization subject to utility constraints (Problem 2) in terms of user-wise MSEs. This stands in some contrast to typical direct formulations in the SINRs (or rates) but from a coding perspective appears to be advantageous for parallel BC systems such as OFDM.

a) Problem 1: Given a general utility function $\psi : \mathbb{R}_+ \mapsto \mathbb{R}$, $\gamma = [\gamma_1, \dots, \gamma_M]^T$ from (5), and non-negative weights $\mathbf{w} \in \mathbb{R}_+^M$ consider the following resource allocation problem:

$$\begin{aligned}\max \quad & \sum_{m\in\mathcal{M}} w_m \psi(\gamma_m(p_{1,1},\dots,p_{M,K})) \\ \text{subj. to} \quad & \sum_{m\in\mathcal{M}} \sum_{k\in\mathcal{K}} p_{m,k} \le \bar{P}.\end{aligned} \tag{P1}$$

For notational simplification we will also consider the somewhat weaker problem where the utility function is given by $\psi(\gamma_m) = \sum_{k\in\mathcal{K}} \psi'(\gamma_{m,k})$ where ψ' is the per-subchannel utility function (in short utility). Clearly, Problem (P1) is in general intractable if no further constraints are put on ψ. Even harder, virtually no general results are known for parallel BC systems. *Restricting our attention to the case* $K = 1$ let us first collect some known classes for which (P1) can be efficiently solved.

The log-convexity class (Stanczak/Wiczanowski/Boche)[1]**:** Using (4) Problem (P1) can be rewritten in terms of SINRs. Denoting this map by $g : \beta \hookrightarrow \gamma$ (sub-indices omitted) it was shown based on the Perron-Frobenius Theory in [1] that the class of utilities

$$\psi_s := \left\{\psi' : \mathbb{R}_{++} \mapsto \mathbb{R}\,, (\psi' \circ g)^{-1} \text{ is log-convex}\right\}$$

results in convex problem formulations. This class of functions includes important utilities like the approximation of rates $R \approx \log(\beta)$ at high SINR and different fairness classes of the form $\psi(\beta) = \beta^{\bar{\alpha}}/\bar{\alpha}, \bar{\alpha} \le 0$, introduced in [4]. However, a considerable disadvantage of ψ_s is that users are never switched of completely since

$$\lim_{\beta\to 0} \psi'(\beta) = -\infty \tag{6}$$

for all $\psi'(\beta) \in \psi_s$. Furthermore, note that *even for the linear function* $\psi'(\gamma) = \gamma$ *the composition* $(\psi' \circ g)^{-1}$ *is not log-convex* so that the simple complementary MSE maximization is not included in this framework.

A similar class can be deduced from [5]. There a convex formulation is obtained by substituting the power in the logarithmic domain and by representing it as posinomials. This allows also a straight forward generalization to parallel BC systems.

b) Problem 2: From an operator's perspective it is not always desirable to maximize utilities for constraint resources. The dual problem formulation of minimizing the needed resources for given QoS or utility requests has the same relevance. Thus, we are also interested in the complementary problem:

$$\begin{aligned}\min \quad & \sum_{m\in\mathcal{M}\,,k\in\mathcal{K}} p_{m,k} \\ \text{subj. to} \quad & \psi(\gamma_m(p_{1,1},\dots,p_{M,K}) \ge \bar{\psi}_m \; \forall m \in \mathcal{M}\end{aligned} \tag{P2}$$

For both problems we derive a new solvability criteria by using a MSE based utility representation instead of the SINR. This allows us to overcome some shortages of the existing approaches and extend the known class of utilities for which convex problem formulations can be achieved.

IV. Utility optimization based on MSEs

A. Multiuser MSE Region

In this section we study the achievable complementary MSE region and derive present results concerning convexity.

First we consider an arbitrary subchannel k. Assume a fixed sum power budget $\bar{P}_k$ for subchannel k. Solving (3) for the power of user m we obtain

$$p_{m,k} = \gamma_{m,k}\left(\bar{P}_k + \frac{1}{g_{m,k}}\right).$$

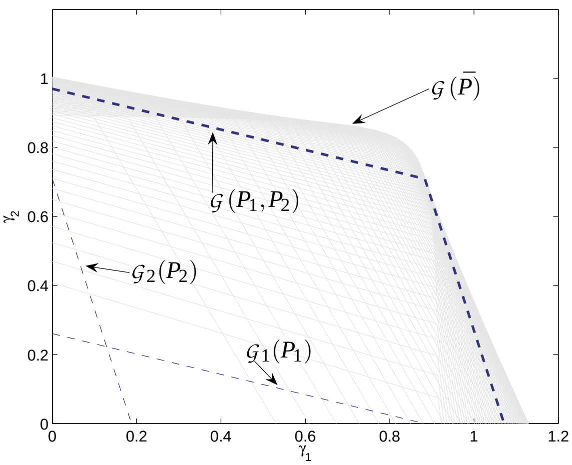

Fig. 1. Illustration of the different complementary MSE regions for a specific choice of P_1 and P_2 with $P_1 + P_2 = \bar{P}$.

Summation over m yields for each subchannel k

$$\sum_{m\in\mathcal{M}} \gamma_{m,k}\left(\bar{P}_k + \frac{1}{g_{m,k}}\right) \le \bar{P}_k$$

and thus the set of achievable complementary MSEs on subchannel k for a fixed sum power $\bar{P}_k$ can be written as

$$\mathcal{G}_k(\bar{P}_k) = \left\{\gamma : \sum_{m\in\mathcal{M}} \gamma_{m,k}\left(1 + \frac{1}{g_{m,k}\bar{P}_k}\right) \le 1\right\}. \tag{7}$$

Clearly (7) is the intersection of a halfspace with $\mathbb{R}_+^M$ and thus a convex set. Moreover, the region is an M-dimensional simplex. The set of achievable complementary sum-MSEs for a fixed power allocation among subchannels can be written as

$$\mathcal{G}(\bar{P}_1,\ldots,\bar{P}_K) = \sum_{k\in\mathcal{K}} \mathcal{G}_k(\bar{P}_k), \tag{8}$$

where the summation of sets is defined as $\mathcal{X} = \sum_n \mathcal{X}_n = \{\mathbf{x} : \mathbf{x} = \sum_n \mathbf{x}_n, \mathbf{x}_n \in \mathcal{X}_n\}$. Obviously, (8) is a polytope and thus also convex. Taking into account the power allocation among subchannels, the achievable complementary MSE region under a sum power constraint is given by

$$\mathcal{G}(\bar{P}) = \bigcup_{P_1,\ldots,P_K : \sum_{k\in\mathcal{K}} P_k = \bar{P}} \mathcal{G}(P_1,\ldots,P_K). \tag{9}$$

Note that all regions equal their dual MAC counterparts based on the following duality theorem explored first by [6]

Theorem 1: If duality holds for SINR then it holds for MSE as well.

Proof: Both SINR and MSE can be achieved if and only if there is some power allocation for an optimal receive filter. Hence, for any feasible SINR, MSE the bijections (3),(4) hold. ■

The relation between $\mathcal{G}(\bar{P})$, $\mathcal{G}(P_1,P_2)$ and $\mathcal{G}_k(P_k)$ is illustrated exemplarily in Figure 1 where it can be seen that the shaded region is non-convex as discussed next.

B. Fixed Power Budgets per Subchannel

A direct consequence of (7) is that Problem (P1) can be recast in the form

$$\begin{aligned} &\max \sum_{m\in\mathcal{M}} w_m \sum_{k\in\mathcal{K}} \psi'(\gamma_{m,k}) \\ &\text{subj. to } \gamma \in \mathcal{G}(\bar{P}_1,\ldots,\bar{P}_M) \end{aligned} \tag{10}$$

or

$$\begin{aligned} &\max \sum_{m\in\mathcal{M}} w_m \psi(\gamma_m) \\ &\text{subj. to } \gamma \in \mathcal{G}(\bar{P}_1,\ldots,\bar{P}_M) \end{aligned} \tag{11}$$

both being convex problems since $\mathcal{G}(\bar{P}_1,\ldots,\bar{P}_K)$ given in (8) is a polytope if ψ is concave. Equivalently the set of utilities

$$\Psi(\bar{P}_1,\ldots,\bar{P}_K) = \Big\{(\psi(\gamma_1),\ldots,\psi(\gamma_M)) : \psi \text{ concave}, \\ \gamma \in \mathcal{G}(\bar{P}_1,\ldots,\bar{P}_M)\Big\}$$

is a convex set.This means that Problem (P1) can be efficiently solved. Furthermore, since the only requirement on ψ' is concavity, assignments with users switched off on subcarriers can be optimal. This cannot happen for the log-convex class because of (6). Interestingly, we have already seen that $\psi'(\gamma) = \gamma$ is not in ψ_s and vice versa the log-convex utility $\psi'(\beta) = \log(\beta)$ is not concave in γ under the map g^{-1}. Thus neither of both classes are subsets of the other class.

C. Sum Power Budget over all Subchannels

The power allocation among subchannels has been assumed to be constant so far. Concerning the structure of (9) the situation turns out to be more complicated than in (8).

In the previous subsection it was shown that for any fixed power allocation the set $\mathcal{G}(P_1,..,P_M)$ is a polytope and thus convex. Surprisingly it turns out that this property gets lost, if power allocation is taken into account.

Lemma 1: The complementary MSE region under a sum power constraint $\mathcal{G}(\bar{P})$ defined in (9) is not necessarily a convex set.

Proof: We study a pathological channel assuming convexity of $\mathcal{G}(\bar{P})$ and show that this leads to a contradiction. Consider a system with $K = M = 2$, normalized sum power $\bar{P} = 1$ and

$$\mathbf{G} = \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} = \begin{pmatrix} 100 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $p^{(1)}$ and $p^{(2)}$ be power allocations

$$\mathbf{P}^{(1)} = \begin{pmatrix} p_{1,1}^{(1)} & p_{1,2}^{(1)} \\ p_{2,1}^{(1)} & p_{2,2}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 0.5 \\ 0 & 0.5 \end{pmatrix}$$

and

$$\mathbf{P}^{(2)} = \begin{pmatrix} p_{1,1}^{(2)} & p_{1,2}^{(2)} \\ p_{2,1}^{(2)} & p_{2,2}^{(2)} \end{pmatrix} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.9 \end{pmatrix}.$$

This leads to

$$\gamma^{(1)} = \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix} \quad \text{and} \quad \gamma^{(2)} = \begin{pmatrix} 0.91 \\ 0.47 \end{pmatrix}.$$

Due to convexity of $\mathcal{G}(\bar{P})$ the tuple

$$\gamma^* = \frac{1}{2}(\gamma^{(1)} + \gamma^{(2)}) = \begin{pmatrix} 0.45 \\ 0.57 \end{pmatrix}$$

must be achievable with sum power $\bar{P} \leq 1$.

The set $\mathcal{G}(\bar{P})$ can be upper bounded by

$$\mathcal{G}(\bar{P}) \subseteq \bigcup_{0 \leq p \leq 1} \left\{ \gamma \in \mathbb{R}_+^2 : \gamma_2 \leq -\gamma_1 a(p) + b(p) \right\}$$

where

$$a(p) = \frac{g_{2,1}p}{g_{2,1}p+1} \frac{g_{1,1}p+1}{g_{1,1}p}$$

and

$$b(p) = \frac{g_{2,1}p}{g_{2,1}p+1} + \frac{g_{2,2}(1-p)}{g_{2,2}(1-p)+1}.$$

This is illustrated in the upper part of Figure 2. The function

$$\gamma_2(p) = -\gamma_1 a(p) + b(p)$$

is concave in $p \in [0,1]$ and thus using and upper bound on the achievable γ_2 can be found for any given value of γ_1. Setting $\gamma_1 = \gamma_1^*$ and solving

$$\tilde{\gamma}_2 = \max_{p \in [0,1]} -\gamma_1^* a(p) + b(p)$$

leads to (see bottom of Figure 2)

$$\tilde{\gamma}_2 < \gamma_2^*$$

which is a contradiction, since γ_2^* must be achievable due to the convexity of $\mathcal{G}(\bar{P})$. Thus, $\mathcal{G}(\bar{P})$ is not a convex set. ■

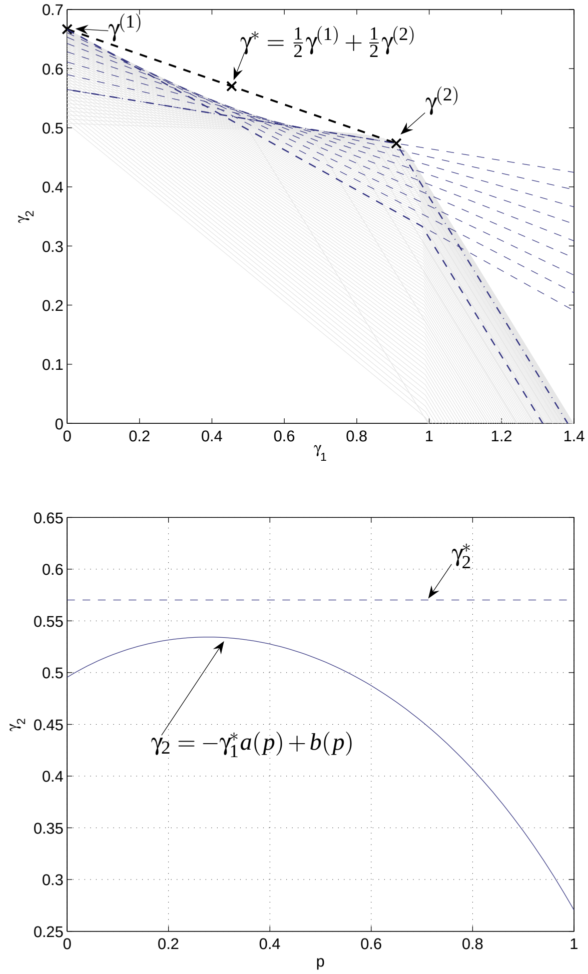

Fig. 2. Illustration of the proof of Lemma 1: Achievable region and convex-combination γ^* (top). Gap between γ_2^* and $\gamma_2 = -\gamma_1^* a(p) + b(p)$ (bottom).

D. Consequences for the MIMO BC MSE Region

An interesting consequence of Lemma 1 is the following corollary.

Corollary 1: The region of achievable individual MSEs of the MIMO BC is not necessarily a convex set.

Proof: This follows from the fact that any system of parallel BCs can be written as a block-diagonal MIMO BC. ■

Note that in [7, Thm. 4] it is claimed that the 2-user MIMO MAC MSE region is a convex set. A direct consequence of the proof techniques used in [7, Thm. 4,Thm. 5] would be the convexity of the 2-user MIMO BC MSE region. This, however contradicts Corollary 1. The reason for that is that in [7] only tuples with the same sum-MSE are considered, which is not sufficient for proving convexity of the whole region.

Unfortunately, it is not clear up to now in which special cases convexity can be guaranteed.

E. Convexity under the Square Root Law

While in the case of per subcarrier constraints convexity is achieved in general we have seen that this does not hold for the region $\mathcal{G}(\bar{P})$. However under suitable re-parameterization the region becomes convex.

Lemma 2: Under the transformation

$$\psi'(\gamma) = \gamma^{1/\alpha} \quad \alpha \geq 2,\ \gamma \geq 0, \tag{12}$$

the set

$$\Psi'(\bar{P}) = \left\{ \sum_k \psi'(\gamma_{1,k}), \ldots, \sum_k \psi'(\gamma_{M,k}),\ \gamma \in \mathcal{G}(\bar{P}) \right\} \tag{13}$$

is a convex set.

Proof: The mapping $\psi'(\gamma)$ is:

- normalized, i.e. $\gamma = 0 \rightarrow \psi'(\gamma) = 0$
- concave and
- strictly monotonously increasing

Thus, an inverse function $\gamma = \phi(x) = (\psi')^{-1}(x)$ exists and $\Psi'(\bar{P})$ is characterized by the following set of equations:

$$\begin{aligned} \sum_{m \in \mathcal{M}} \phi(x) \left(1 + \frac{1}{g_{m,k}P_k}\right) &\leq 1 \quad \forall k \in \mathcal{K} \\ \sum_{m \in \mathcal{K}} P_k &\leq \bar{P} \end{aligned} \tag{14}$$

A sufficient condition for the set $\Psi'(\bar{P})$ being convex is that the expression

$$f(x,P) = \phi(x)\left(1 + \frac{1}{gP}\right)$$

is jointly convex in (x,P). Now consider the Hessian of $f(x,P)$ given by

$$\mathbf{H}_{f(x,P)} = \begin{pmatrix} \frac{\partial^2\phi(x)}{\partial x^2}(1+\frac{1}{gP}) & -\frac{\partial\phi(x)}{\partial x}\frac{1}{gP^2} \\ -\frac{\partial\phi(x)}{\partial x}\frac{1}{gP^2} & \phi(x)\frac{2}{gP^3} \end{pmatrix}.$$

Due to the assumptions made on $\psi(\gamma)$ we have

$$\operatorname{tr}\left(\mathbf{H}_{f(x,P)}\right) \geq 0. \tag{15}$$

Hence a sufficient condition for positive-semi-definiteness of $\mathbf{H}_{f(x,P)}$ and thus convexity of the problem is

$$\det\left(\mathbf{H}_{f(x,P)}\right) \geq 0. \tag{16}$$

Using (12), after some simple manipulations (16) can be rewritten

$$\det\left(\mathbf{H}_{f(x,P)}\right) = \underbrace{\alpha^2 x^{2\alpha-2}\frac{1}{g^2P^4}}_{=:A}\underbrace{\left(2(1+gP)(1-\frac{1}{\alpha})-1\right)}_{=:B}$$

Obviously $A \geq 0$ independent of α and $B \geq 0$ is true if

$$\alpha \geq \frac{2(1+gP)}{2(1+gP)-1}.$$

and therefore

$$\alpha \geq 2$$

which concludes the proof. ■

Therefore, setting $\alpha \geq 2$ is sufficient to turn the original region into a convex one. This is referred to as the square root law.

Corollary 2 (high SINR): For high powers the expression converges to

$$\lim_{P\to\infty}\frac{2(1+gP)}{2(1+gP)-1} = 1$$

allowing even linear utilities in approximation. Hence, also the standard MSE optimization can be solved with convex optimization tools at high SINR.

This characteristic does not hold for low SINR. However, we make an interesting observation in terms of maximizing the throughput. In general the rate $R = \log(1+\beta)$ as utility function belongs neither to the log convex class in [1], nor is a concave utility function in terms of the CMSE. Complementing the known result, that at high SINR the ergodic rate belongs to the log convex class, we get the following result for low SINR: For low SINR maximizing the rate for individual power constraints and maximizing the square root of the ergodic rate for a sum power constraint, respectively, can be approximated by a convex optimization problem. This relies on the following observations. For individual power constraints only ψ has to be concave. Using the approximation of the rate for low SINR

$$R = \log(1+\beta) = -\log(1-\gamma) \approx \gamma \tag{17}$$

results in a concave mapping.

For the sum power constraint we can approximate

$$R^{1/2}(\gamma) = (-\log(1-\sqrt{\gamma}^2))^{1/2} \approx \gamma \tag{18}$$

Therefore $R^{1/2} \in \psi_M$, where ψ_M will be introduced in the following section.

F. A New Class of Utility Functions

The latter derivations open up a way for efficiently solving the problem

$$\begin{aligned} \max \quad & \sum_{m\in\mathcal{M}} w_m \sum_{k\in\mathcal{K}} \psi'(\gamma_{m,k}) \\ \text{subj. to} \quad & \sum_{m\in\mathcal{M}} \gamma_{m,k}\left(1+\frac{1}{g_{m,k}P_K}\right) \forall k \in \mathcal{K} \\ & \sum_{k\in\mathcal{K}} P_k \leq \bar{P} \end{aligned} \tag{19}$$

based on the following interpretation: Define $g'(\gamma) = \gamma^2$. If the utility ψ' is such that

$$\psi_M = \left\{\psi' : \mathbb{R}_+ \mapsto \mathbb{R}\,, \psi' \circ g' \text{ is concave}\right\}, \tag{20}$$

we can recast the problem

$$\begin{aligned} \max \quad & \sum_{m\in\mathcal{M}} w_m \sum_{k\in\mathcal{K}} \psi'(x_{m,k}^2) \\ \text{subj. to} \quad & \sum_{m\in\mathcal{M}} x_{m,k}^2\left(1+\frac{1}{g_{m,k}P_k}\right) \forall k \in \mathcal{K} \\ & \sum_{k\in\mathcal{K}} P_k \leq \bar{P} \end{aligned} \tag{21}$$

which is convex and thus can be solved with standard convex optimization tools. This result leads to the following conclusion: Problem (21) can be reformulated as

$$\max \sum_{m\in\mathcal{M}} w_m q_m \quad \text{subj. to } \mathbf{q} \in, \Psi'(\bar{P}) \tag{22}$$

which is convex in

$$q_m := \sum_{k\in\mathcal{K}} \psi'(\gamma_{m,k}). \tag{23}$$

Using the former definition we get the following Theorem:

Theorem 2: Suppose that the utility $\psi'(\gamma)$ belongs to ψ_M, then then the Karush-Kuhn-Tucker conditions are necessary and sufficient for the solution of Problem (P1) in p.

Proof: The proof relies on the fact that Ψ' is a convex set. Suppose that there are two power allocations $\mathbf{p}^{(1)}, \mathbf{p}^{(2)}$ such that there is no local ascent direction. By assumption both power allocations lead to a different value of the objective function. Since the objective function is linear in $\mathbf{q}$ according to (23) this is equivalent to saying that there were two points $\mathbf{q}^{(1)}, \mathbf{q}^{(2)}$ where the hyperplane defined by the objective function is supported. Furthermore, there can be no feasible convex combination between $\mathbf{q}^{(1)}, \mathbf{q}^{(2)}$ leading to an ascent of the objective function. However, this contradicts the fact that Ψ' is a convex set. ■

It is noted here that utility functions do not have to comply with the square root law. More general, each concave utility function ψ' with its inverse ϕ results in a convex optimization problem if (15) and (16) hold.

V. MINIMUM SUM POWER UNDER MSE CONSTRAINTS

To minimize the sum power for given MSE constraints we can now recast Problem (P2) in following form:

$$\begin{aligned} \min \quad & \sum_{k\in\mathcal{K}} P_k \\ \text{subj. to} \quad & \sum_{m\in\mathcal{M}} \gamma_{m,k}\left(1+\frac{1}{g_{m,k}P_k}\right) \leq 1 \quad \forall k\in\mathcal{K} \\ & \sum_{k\in\mathcal{K}} \psi'(\gamma_{m,k}) \geq \bar{\psi}_m \quad \forall m\in\mathcal{M} \end{aligned} \tag{24}$$

It is easy to see that the derivations made for Problem (P1) equivalently hold for the sum-power minimization problem.

Theorem 3: Suppose that the utility $\psi'(\gamma)$ belongs to ψ_M, then then the Karush-Kuhn-Tucker conditions are necessary and sufficient for the solution of Problem (P2).

Proof: Similar to the proof of *Theorem* 2. ∎

After showing that the problems can both, either be converted into a convex representation or that only one global optimum exists for the original formulation for the presented utility class, designing suitable algorithms is the next challenge. This will be tackled in an companion paper.

VI. CONCLUSION

In this paper we studied the convexity of the MSE region for parallel broadcast channels. We formulated two complementary optimization problems based on MSE dependent utility functions: The maximization of the weighted sum of user utilities at a given sum power constraint and the minimization of the sum power for given minimum utility requirements. For both problems a new class of utility functions was defined that always allows a convex problem formulation. Therefore the usage of standard tools from convex optimization is possible and convergence to the global optimum in polynomial time can be guaranteed. By introducing the square root criteria we found a straight forward rule to check if a utility functions belongs to our class. This set of optimization metrics allows also switching off users at the optimum and therefore presents a valuable extension to the known class of log-convex utilities.

VII. ACKNOWLEDGMENT

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