A Short-Term Throughput Measure for Communications with ARQ Protocols

Anastasios Giovanidis[1], Gerhard Wunder[1] and Holger Boche[1,2]
[1]Fraunhofer Institute for Telecommunications, Heinrich Hertz Institute, MCI, Einsteinufer 37, 10587 Berlin, Germany
[2]Heinrich Hertz Chair for Mobile Communications, Faculty EECS, TU Berlin, Einsteinufer 25, 10587 Berlin, Germany
{giovanidis, wunder}@hhi.de, {holger.boche}@mk.tu-berlin.de

Abstract

The renewal-reward theorem has been widely used in the literature to provide the expected throughput of ARQ systems when the inter-arrival intervals between consecutive message acknowledgements (ACKs) are iid random variables. In cases however when the channel error probability may be allowed to vary with time the above approach cannot accurately estimate the desired throughput time-average. For such scenarios we propose a 'short-term' measure for the expected throughput of ARQ systems up to some predefined number r of received ACKs and explicitly show that it converges asymptotically to the 'long-term' renewal-reward result when $r \rightarrow \infty$. A Maximum Zero-outage short-term Throughput (MZT_{s-t}) metric is further introduced. Its value is shown to be greater than the respective value (MZT_{l-t}) for the 'long-term' case. We conclude that use of the renewal-reward theorem in cases of varying error probabilities may underestimate the maximum possible transmission rate and the short-term measure is more appropriate.

1 Introduction

In wireless packet communications, automatic repeat request (ARQ) protocols are used to overcome transmission errors, occuring due to the fading nature of the channel, use of finite-length codewords and the presence of interference. On the other hand support of specific quality of service raises delay constraints. In such practical systems existing measures, such as *ergodic capacity*, *ϵ-capacity* or *delay-limited capacity* either cannot quantify error-free communications or are overly conservative, as was rather interestingly explained in [1]. For this reason in [1] and [2] the focus concerning channels with retransmissions is on throughput ('goodput') rather than transmission rate and the aim is its maximization given some probability of error for the channel [1]. Application of the *'renewal-reward' theorem* [3] can provide the long-term throughput time-average of communication systems incorporating retransmissions, as suggested in [4] and has been vastly used in the current literature as a measure of system performance.

Consider now a scenario where multiple access terminals within a cell communicate with a base station. For each user a queue can temporarily buffer the amount of randomly arriving data which cannot be immediately served due to limited resources, while a simple Stop-and-Wait ARQ protocol with success probability per trial equal to q is implemented for correction of possible erroneous transmissions. In such systems a fundamental problem is the optimal allocation of resources among the users with the aim to maximize throughput and minimize latency. In recent literature ([5], [6] and references therein) cross-layer approaches have been suggested where information from medium access control (MAC) and the physical (PHY) layer are combined to optimally adapt the users' power and rate to the instantaneous channel quality taking into account QoS parameters such as current queue lengths and maximum acceptable delay. Scheduling decisions are put into action after each message retransmission circle has successfully (or unsuccessfully in the case of truncated ARQ protocols) been finalized and will affect the next waiting buffered packet to be sent. In such scenarios the channel's probability of error is bound to change each time a new decision is made over the values of transmission rate and power. Other possible cases for a change of the transmission error probability with time is when the acknowledgement history is used to optimize decisions over future retransmissions, as suggested in [7], [8], or when bursty errors occur often in an unpredicted fashion.

In such dynamic situations as mentioned above, estimation of the throughput in the long-run based on the renewal-reward theorem is not any more applicable. This is due to the simple fact that the inter-arrival times between consecutive ACKs are not any more identically distributed. In such scenarios the assumption that the renewal process probabilistically starts anew after each packet reception [3] does not hold. A discussion over the renewal-reward theorem and its use in the recent communications literature related to ARQ is provided in Section 2.

This research is supported by the German Federal Ministry of Education and Research as part of the ScaleNet project 01BU566.

In the current work we suggest and investigate the use of a new measure to provide estimates of the expected system throughput in such cases where the renewal-reward theorem cannot be applied. This measure is the expected value up to some predefined *finite* number of acknowledgements (ACKs), say r, for which the error probability of the channel is expected to remain constant and afterwards probably changes and is introduced in Section 3. The simplest case for $r = 1$ is the expected throughput up to first ACK and referes to a scenario where scheduling is dynamically bound to change after each correct packet reception, called the *'short-term ARQ throughput'*. In what follows explicit expressions are provided and the convergence to the 'long-term ARQ throughput' (renewal-reward) is proved, as expected, for $r \to \infty$. The short-term value is greater than the long-term one, as a direct consequence of *optionally stopping* after the r-th success [9].

Following [1] we define in Section 4 the *Maximum Zero-Outage short-term Throughput* (MZT_{s-t}), derive its solution for the special case of Rayleigh fading channel statistics and prove that the transmission rate to achieve it is greater than the argument $R_{l-t}^{\max} = \mathcal{W}(\rho)$ obtained in the long-term case. We thus conclude that the renewal-reward theorem may actually underestimate the maximum rate that can be transmitted through the channel, when we are interested in short-term gains. The comparison between the suggested measure and the throughput using the renewal-reward theorem is further illustrated in plots. Section 5 contains the conclusions of our work. Proofs of all propositions in the main text can be found in the Appendix.

2 Some Renewal Theory Results

Based on renewal-reward theory retransmissions can be understood as a renewal process. In this case a renewal event occurs when an ACK is fed back after a certain non-negative number of consecutive NACKs due to erroneous packet transmission. The inter-renewal time is a random variable denoted as N, with uncountable discrete state-space $\mathcal{N}_+ = \{1, 2, \ldots\}$ the number of efforts until correct packet reception. The time at which a renewal occurs is further a discrete process (arrival process) given as the sum of the inter-renewal intervals up to the current renewal epoch. $S_k = N_1 + \ldots + N_k$ is the time of occurence for the k-th renewal. With the above discribed renewal processes a *reward function* $R(t)$ is further introduced that models a rate at which the renewal process accumulates a reward. The value of $R(t)$ depends on the *age* of the process t and the *duration* of the inter-renewal interval containing time t. Here the reward may be viewed as a rate gain each time a message is correctly received. Then if the constant transmission rate of the system is R and $N_k = n$ is the inter-renewal interval for $S_{k-1} < t \leq S_k$, the actual rate gain between the $k-1$-th and k-th ACK (throughput or 'goodput') equals $\frac{R}{n}$ forming a reward function with the aforementioned properties. The following two theorems [3] can provide the long-term system throughput of a communications system using ARQ protocols. The first one focuses on limiting behaviors of time averages while the second one on ensemble averages. Variables t and τ take values on the non-negative discrete time axis since the communications systems are usually considered time-slotted.

Theorem 1 *Let $R(t)$ be a renewal function providing the rate gain of the ARQ system at time t, with expected inter-renewal time $E[N]$. If $E[N] < \infty$ or $E[R_k] < \infty$, then with probability 1*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t} R(\tau) = \frac{E[R_k]}{E[N]} \tag{1}$$

where

$$R_k = (S_{k-1} - S_k) \cdot \frac{R}{n} = n \cdot \frac{R}{n} = R \quad \& \quad N_k = n \tag{2}$$

Then the long-term system throughput equals

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t} R(\tau) = \frac{R}{E[N]} \tag{3}$$

Theorem 2 *Let $R(t)$ be a renewal function providing the rate gain of the ARQ system at renewal time t, with expected inter-renewal time $E[N]$. If renewals occur only at discrete-time epochs (meaning $t \in \mathcal{N}$) then with probability 1*

$$\lim_{t\to\infty} E[R(t)] = \frac{E[R_k]}{E[N]} = \frac{R}{E[N]} \tag{4}$$

Observe here the *equality of the limiting behavior for the time and ensemble average of the renewal process*. The value of the above average depends solely on the distribution of the inter-arrival times and silently assumes that inter-arrival random variables are independent identically distributed throughout the renewal process.

In wireless communications, transmission errors can be described as outages having probability of occurence $\mathcal{P}_{out}(\rho, R)$ - a function of power ρ and rate R.

$$\mathcal{P}_{out}(\rho, R) \quad = \quad \mathcal{P}\left(\log\left(1 + |H(f)|^2 \rho\right) \leq R\right) \tag{5}$$

Given the above expression, and denoting by $\mathcal{E}_i$ the event of outage at trial i, the probability that a codeword will need n efforts until it is correctly received is [1]

$$\mathcal{P}(N = n) = \mathcal{P}\left(\bigcap_{i=1}^{n-1} \mathcal{E}_i\right)\left[1 - \mathcal{P}\left(\mathcal{E}_n | \bigcap_{i=1}^{n-1} \mathcal{E}_i\right)\right] \tag{6}$$

and the throughput of the communications system can be provided by the previous theorems

$$\eta = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t} R(\tau) = \lim_{t\to\infty} E[R(t)] = \frac{R}{E[N]} \tag{7}$$

If the error probability per trial remains constant, the retransmissions resemble Bernoulli trials and the throughput merely equals $R\left(1-\mathcal{P}_{out}\right)$. In this case N is *geometrically* distributed. In what follows the above expression will be called the 'long-term ARQ throughput'.

Using the aforementioned framework, the authors in [2] have derived explicit expressions for the throughput of different Hybrid-ARQ protocols operating in channels with Gaussian noise and interference, whereas in [1] and assuming equal success probabilities per trial the authors have introduced a new metric, namely the *Maximum Zero-Outage Throughput* (MZT) which equals the maximum achievable throughput in fading channels with errors. For Rayleigh fading an explicit solution has been derived

$$\begin{aligned} MZT\left(\rho\right) &= \sup_{R} R\left[1-\mathcal{P}_{out}\left(\rho,R\right)\right] && (8)\\ &\overset{Rayleigh}{=} \sup_{R} R\cdot\exp\left(-\rho^{-1}\left(e^{R}-1\right)\right) && (9)\end{aligned}$$

The argument maximizing expression (9) equals $R_{l-t}^{\max}=\mathcal{W}\left(\rho\right)$ where $W\left(z\right)$ solves the equation $\mathcal{W}\left(z\right)\cdot e^{\mathcal{W}(z)}=z$ and is called the Lambert $\mathcal{W}$ function [10].

3 Short-term Measure

In scenarios presented in the introduction, where the channel error probability varies with time, the assumption of identically distributed inter-renewal intervals does not hold. In such cases the limiting time-average or ensemble-average behavior of the reward cannot actually be used to provide a measure of system throughput. In the following we suggest the use of short-term measures to estimate throughput performance. We assume that the channel error probability remains constant up to some predefined finite number of ACKs (renewals), say r, and then probably changes. Given a scenario where after each ACK the scheduler changes the values of allocated power or transmission rate for the observed user, its *short-term throughput* can be derived as the ensemble average of the reward up to next ACK, namely $E\left[\frac{R}{N}\right]$. In what follows we consider geometrically distributed inter-arrival times between renewal epochs, however the analysis can as well be applied to more complicated distributions. $N^{(r)}$ resembles the random variable for the inter-arrival period between r consecutive ACKs and has in the current text a *negative binomial* distribution. As the number of r is allowed to increase, tending to infinity, the convergence to the long-term throughput results directly from the renewal-reward theorem for ensemble averages given as Theorem 2. The probability of success equals q and of failure $p=1-q$.

Proposition 1 *For an ARQ protocol with geometrically distributed inter-arrival intervals, the expected value of throughput (reward) until a single correct message reception (first ACK) equals*

$$T_{s-t}\left(\rho,R\right)=E\left[\frac{R}{N}\right]=R\frac{p-1}{p}\log\left(1-p\right) \qquad (10)$$

This is called the 'short-term' system throughput. The expected value until r successful receptions is

$$T_{s-t}^{(r)}\left(\rho,R\right)=R\cdot E\left[\frac{r}{N^{(r)}}\right]$$

$$=R\cdot\left[\sum_{j=1}^{r-1}\frac{-r}{r-j}\left(\frac{p-1}{p}\right)^{j}+\left(\frac{p-1}{p}\right)^{r}r\log\left(1-p\right)\right] \qquad (11)$$

As the number of accepted messages approaches infinity the expected value of throughput has the following limit

$$T_{l-t}\left(\rho,R\right)=\frac{R}{E\left[N\right]}=R\left(1-p\right) \qquad (12)$$

This provides the 'long-term' system throughput, equal to that using the 'renewal-reward' theorem.

The behavior of the expected value of throughput $T_{s-t}^{(r)}$ as the predefined number r of acknowledgements increases is illustrated in figure 1 for different values of error probabilities. It is clearly shown that the short-term measure converges asymptotically to its long-term value as r increases. Furthermore, it can be observed that the results using renewal-reward may differ much from the short-term case. For $r=1$ the following Corollary explicitly emphasizes the comparison between the two measures.

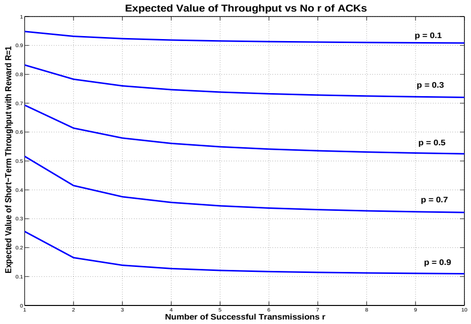

Abbildung 1: Expected throughput depends on error probability as well as number r of successful retransmissions (Optional Stopping). When $r\to\infty$ the expected throughput tends to $R\left(1-p\right)$ which is the long-term throughput.

Corollary 1 *The short-term throughput has a greater value compared to the long-term one.*

$$T_{s-t}\geq T_{l-t}\Leftrightarrow E\left[\frac{R}{N}\right]\geq\frac{R}{E\left[N\right]} \qquad (13)$$

Proof. This comes directly from Jensen's inequality since

$$E\left[g\left(x\right)\right]\geq g\left(E\left[x\right]\right) \qquad (14)$$

for $g(x)$ convex function of x. In our case $g(x)=R/x$ (convex) and (13) holds. ■

The conclusion that the actual expected average per stage until first ACK (or up to $r < \infty$ acknowledgements in general) $E\left[\frac{R}{N}\right]$ has a greater value compared to the renewal-reward result $\frac{R}{E[N]}$ may at a first glance seem a bit unexpected and possibly paradoxal. In an effort to provide some intuition and explain the results, we notice that the gains in short-term actually are a result of *stopping* the observation of the renewal process at a desired moment, namely when exactly r ACKs have been received. In such cases the probability of occurence of very long NACK runs is very small. If the process, however, is allowed to evolve extremely long runs are expected to happen. It can actually be found in Feller [9] that using the law of iterated logarithm, the length of the longest appearing run after t efforts in a sequence of Bernoulli trials behaves with probability 1 as follows

$$\lim_{t\to\infty} \sup \frac{N_t^{longest}}{\log_{1/p} t} = 1 \tag{15}$$

The distribution of the longest run for a process generated by fair coin tossing is known as the Erdös-Rényi law [11]. Relevant investigations can be found in the literature e.g. [12] and references therein. The length of the longest run of NACKs then asymptotically tends to infinity as the number of trials $m \to \infty$. Occurence of such rare extrema has an averaging out effect for the strong law of large numbers to hold. By stopping the process after a certain finite number r such extrema are not probable to happen.

4 Max Zero-Outage Throughput

When practical systems are under study, we are often interested in estimating which is the maximum achievable throughput in future steps and aim to adapt the systems' operation in order to optimize their performance and stay always near the maximum value. In our case the adaptation takes place in terms of estimating and adopting the optimal transmission rate. Rather interestingly use of different measures provides different results. This actually implies that the use of appropriate throughput measures in different communication scenarios can be rather critical in terms of performance optimization. The *Maximum Zero-Outage short-term Throughput* (MZT_{s-t}) is defined as

$$T_{s-t}^{\max} = \sup_R E\left[\frac{R}{N}\right] = \sup_R R\frac{p-1}{p}\log(1-p) \tag{16}$$

Assumption of Rayleigh fading statistics simplifies analysis and can provide results which are easier to compare and illustrate.

Proposition 2 *The transmission rate that maximizes the short-term throughput $R_{s-t}^{\max} = \arg\max_R T_{s-t}$, for Rayleigh fading channels and small values of $\frac{1-e^R}{\rho}$ is*

$$R_{s-t}^{\max} \approx \mathcal{W}\left((2\rho+1)\cdot e\right) - 1 \tag{17}$$

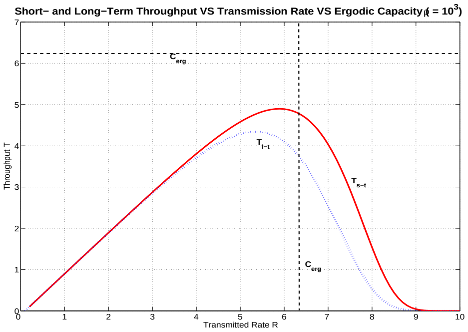

Abbildung 2: Throughput vs rate curves for average power $\rho = 10^3$. Comparison between short-term, long-term throughput and ergodic capacity $C_{erg} = 6.338$ b/s/Hz

Proposition 3 *For Rayleigh fading channels the transmission rate maximizing the short-term throughput is greater than the argument maximizing the long-term one.*

$$R_{s-t}^{\max} \geq R_{l-t}^{\max} \tag{18}$$

Given specific power $\rho = 10^3$ and assuming Rayleigh fading, the expected throughput vs rate plots are illustrated in figure 2. The doted horizontal and vertical lines denote the value of ergodic capacity. Expected throughput cannot exceed this limit, however the plots support the conclusions in [1] that in systems with retransmissions, transmission rate may actually be allowed to exceed the ergodic limit and probable errors will be corrected with the use of retransmissions. In such cases there is a non-zero probability that transmitted bits at an instantaneous rate higher than the ergodic may be correctly decoded. If an error occurs, retransmissions take place resulting in a lower effective rate. In average throughput is always less than the ergodic capacity. The figure illustrates that the rate achieving MZT_{s-t} is higher than the one that brings MZT_{l-t}. The actual maximum throughput values for short-term and long-term measures exhibit the same behavior. The change of the maximum rate and throughput as power increases is shown in figure 3.

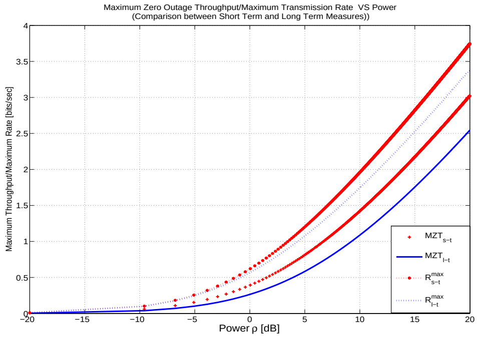

Abbildung 3: Short-term and long-term maximum transmission rate and MZT vs power

5 Conclusions

In cases where the channel error probability may be allowed to change with time, possibly due to application of adaptive transmission techniques, varying resource allocation or fluctuation of the moments of the channel statistics, the renewal-reward approach is not accurate for the estimation of the expected ARQ throughput, since the interarrival times between consecutive acknowledgements are not any more identically distributed. In such scenarios the use of an alternative measure for the expected system throughput was suggested in the current work, refered to as 'short-term ARQ throughput', based on the assumption that the error probability may remain constant up to some predefined number r of received acknowledgements and it is afterwards bound to change. We proved that as $r \to \infty$ the suggested measure asymptotically approaches the 'long-term' throughput. A metric named Maximum Zero-Outage short-term Throughput MZT_{s-t} provides the maximum achievable expected throughput when $r = 1$ which was shown to be greater compared to the long-term case. Comparison plots supported the results of the analysis.

Acknowledgements

The authors would like to thank Dr. Sławomir Stańczak for his helpful comments during the current research work.

Appendix

Proof. [**Proposition 1**] The throughput of the system for this single successful transmission equals $g(n) = R/n$ and this is given as 'reward' due to the occurence of a renewal at the n-th trial. The pdf of the number of efforts up to first ACK is geometrically distributed and given by $f(n) = p^{n-1}q$. The expected value of the reward in a single renewal period (short-term) is

$$\begin{aligned} T_{s-t}(\rho, R) &= E\left[\frac{R}{N}\right] = RE\left[\frac{1}{N}\right] = R\sum_{n=1}^{\infty} g(n)f(n) \\ &= \sum_{n=1}^{\infty} \frac{R}{n} \cdot p^{n-1}q = \frac{Rq}{p}\sum_{n=0}^{\infty} \frac{p^{n+1}}{n+1} \qquad (19) \end{aligned}$$

The series $\sum_{n=0}^{\infty} x^n$ converges uniformly for $|x| < 1$ [13]. Since the integral of the term x^n equals $\frac{x^{n+1}}{n+1}$, from [13],pp.537 the series at the right hand side of (19) also converges uniformly and we can write $\int_0^p \sum_{n=0}^{\infty} x^n dx = \int_0^p \frac{1}{1-x} dx = -\log(1-p)$. Then

$$T_{s-t}(\rho, R) = -R\frac{1-p}{p}\log(1-p)$$

For r successful transmissions (r times ACK) the waiting time until the $r-th$ success is a random variable $N^{(r)}$ with probability $P\left\{N^{(r)} < r\right\} = 0$. $N^{(r)}$ follows a negative binomial distribution [9]

$$P\left\{N^{(r)} = n\right\} = \left(\begin{array}{c} n-1 \\ r-1 \end{array}\right) q^r p^{n-r} = f^{(r)}(n)$$

The reward for r acknowledgements equals $g^{(r)}(n) = \frac{rR}{n}$ and the expected value of the reward

$$\begin{aligned} T_{s-t}^{(r)} &= E\left[\frac{rR}{n}\right] = \sum_{n=r}^{\infty} g^{(r)}(n) \cdot f^{(r)}(n) \\ &= R\sum_{n=r}^{\infty} \frac{r}{n} \cdot \left(\begin{array}{c} n-1 \\ r-1 \end{array}\right) q^r p^{n-r} \\ &= R\left(\frac{q}{p}\right)^r \sum_{n=r}^{\infty} \frac{r}{n} \left(\begin{array}{c} b-1 \\ r-1 \end{array}\right) p^b = R\left(\frac{q}{p}\right)^r u(p) \end{aligned}$$

Taking the first derivative of function $u(p)$

$$\begin{aligned} \frac{du(p)}{dp} &= r\sum_{n=r}^{\infty} \left(\begin{array}{c} n-1 \\ r-1 \end{array}\right) p^{n-1} \overset{n-r=\nu}{=} \\ &= rp^{r-1}\sum_{\nu=0}^{\infty} \left(\begin{array}{c} \nu+r-1 \\ \nu \end{array}\right) p^{\nu} = \\ &\overset{(a)}{=} \sum_{\nu=0}^{\infty} rp^{r-1} \left(\begin{array}{c} -r \\ \nu \end{array}\right) (-p)^{\nu} \overset{(b)}{=} r(1-p)^{-r} p^{r-1} \end{aligned}$$

where equality (a) comes from the fact that $\left(\begin{array}{c} \nu+r-1 \\ \nu \end{array}\right) = (-1)^{\nu} \left(\begin{array}{c} -r \\ \nu \end{array}\right)$ and equality (b) from Newton's binomial formula [13]. Then

$$T_{s-t}^{(r)} = R\left(\frac{q}{p}\right)^r \cdot \int_0^p r(1-x)^{-r} x^{r-1} dx$$

The integral can be evaluated by repeated integrations.

$$\begin{aligned} T_{s-t}^{(r)}(\rho, R) &= R\sum_{j=1}^{r-1} (-1)^{j-1} \frac{r}{r-j} \left(\frac{1-p}{p}\right)^j \\ &+ R\left(\frac{p-1}{p}\right)^r r\log(1-p) \qquad (20) \end{aligned}$$

We can rewrite (20) as follows

$$\frac{T_{s-t}^{(r)}}{R} = r\left(\frac{p-1}{p}\right)^r \left[\sum_{j=1}^{r-1} \frac{1}{j-r} \left(\frac{p-1}{p}\right)^{j-r} + \log(1-p)\right]$$

Then for $r+1$ we get the following recursive form

$$\frac{T_{s-t}^{(r+1)}}{R} = \frac{r+1}{r} \cdot \frac{p-1}{p} \left[\frac{T_{s-t}^{(r)}}{R} - 1\right] \qquad (21)$$

Taking limits for both sides of (21) for $r \to \infty$ we obtain the value for $T_{s-t}^{\infty} := \lim_{r\to\infty} T_{s-t}^{(r)}$

$$\frac{T_{s-t}^{(\infty)}}{R} = \frac{p-1}{p} \left[\frac{T_{s-t}^{(\infty)}}{R} - 1\right] \Rightarrow T_{s-t}^{(\infty)} = R(1-p) = T_{l-t}$$

■

Proof. [**Proposition 2**] For Rayleigh fading, eq. (10) yields

$$T_{s-t} = R \cdot \left(1 - \frac{1}{1 - e^{\frac{1-e^R}{\rho}}}\right) \cdot \frac{1-e^R}{\rho}$$

Let us take the first derivative of T_{s-t} w.r.t. R and replace $u = \frac{1-e^R}{\rho}$ to make the form easier to handle

$$\begin{aligned} \frac{dT_{s-t}}{dR} &= \left(1 - \frac{1}{1-e^u}\right) u \\ &+ R\left(1 - \frac{1}{1-e^u}\right)\left(u - \frac{1}{\rho}\right) - Ru\frac{\left(u - \frac{1}{\rho}\right)e^u}{\left(1-e^u\right)^2} \end{aligned}$$

Setting the derivative equal to zero we have

$$\begin{aligned} \left(1-e^u\right)\left[-u - R\left(u - \frac{1}{\rho}\right)\right] - Ru\left(u - \frac{1}{\rho}\right) &= 0 \\ \& \ R &\neq 0 \end{aligned}$$

If we approximate the exponential term by its Taylor expansion until the 2nd power

$$e^u \approx 1 + \frac{u}{1!} + \frac{u^2}{2!}$$

$$\begin{aligned} \left(u + \frac{u^2}{2}\right)\left[u + R\left(u - \frac{1}{\rho}\right)\right] &= Ru\left(u - \frac{1}{\rho}\right) \Rightarrow \\ u^2\left[1 + \frac{1}{2}\left(u + R(u - \frac{1}{\rho})\right)\right] &= 0 \overset{R\neq 0}{\Rightarrow} \\ 2\rho + 1 - e^R - Re^R &= 0 \Rightarrow \\ e^R\left(R+1\right) &= 2\rho + 1 \Rightarrow \\ e^{R+1}\left(R+1\right) &= \left(2\rho+1\right)\cdot e \end{aligned}$$

and we conclude in (17).

■

Proof. [**Proposition 3**] From Proposition 2 and the solution of MZT_{s-t} provided in [1] we are given the following arguments maximizing throughput in the short and long term respectively

$$\begin{aligned} R_{l-t}^{\max} = \mathcal{W}\left((\rho)\right) &:= R_1 \\ R_{s-t}^{\max} = \mathcal{W}\left((2\rho+1)\cdot e\right) - 1 &:= R_2 \end{aligned}$$

in the following we prove that $R_2 \geq R_1$.

$$\left.\begin{array}{l} \left(R_2+1\right)e^{R_2+1} = \left(2\rho+1\right)\cdot e \\ R_1 e^{R_1} = \rho \end{array}\right\} \Rightarrow$$

$$R_2 - R_1 + \log\left(\frac{R_2+1}{R_1}\right) = \log\left(2 + \frac{1}{\rho}\right) \Rightarrow$$

$$e^{R_2 - R_1}\frac{R_2+1}{R_1} = 2 + \frac{1}{\rho}$$

Suppose that $R_2 < R_1$. Then

$$2 + \frac{1}{\rho} = e^{R_2-R_1}\frac{R_2+1}{R_1} < 1 + \frac{1}{R_1} \Rightarrow$$

$$R_1 < \frac{\rho}{1+\rho} \Rightarrow e^{\mathcal{W}(\rho)} < e^{\frac{\rho}{1+\rho}} \Rightarrow$$

$$\mathcal{W}\left(\rho\right)\cdot e^{\mathcal{W}(\rho)} < \frac{\rho}{1+\rho}\cdot e^{\frac{\rho}{1+\rho}} \Rightarrow \rho < \frac{\rho}{1+\rho}\cdot e^{\frac{\rho}{1+\rho}} \Rightarrow$$

$$\log\left(1+\rho\right) - \frac{\rho}{1+\rho} = v\left(\rho\right) < 0$$

But $v\left(0\right) = 0$ and is monotone non-decreasing w.r.t $\rho \geq 0$, since $\frac{dv(\rho)}{d\rho} = \frac{\rho}{(1+\rho)^2} > 0$. Then $v\left(\rho\right) \geq v\left(0\right) = 0$ and we have proved by contradiction that $R_2 \geq R_1$. ■

Literatur

[1] N. Ahmed and R.G. Baraniuk. Throughput Measures for Delay-Constrained Communications in Fading Channels. *41st Annual Allerton Conference on Commun., Control and Computing*, Oct. 2003.

[2] G. Caire and D. Tuninetti. The Throughput of Hybrid-ARQ Protocols for the Gaussian Collision Channel. *IEEE Trans. on Information Theory*, 47, No.5:1971–1988, July 2001.

[3] R. G. Gallager. *Discrete Stochastic Processes*. Kluwer Academic Publishers, 1996.

[4] M. Zorzi and R.R. Rao. On the Use of Renewal Theory in the Analysis of ARQ Protocols. *IEEE Trans. on Communications*, 44, No.9:1077–1081, Sept 1996.

[5] R. A. Berry and E. M. Yeh. Cross-Layer Wireless Resource Allocation. *IEEE Signal Processing Magazine*, 21, no. 5, Sept. 2004.

[6] G. Dimic, N. D. Sidiropoulos, and R. Zhang. Medium Access Control - Physical Cross-Layer Design. *IEEE Signal Processing Magazine*, 4, Sep. 2004.

[7] E. Modiano. An adaptive algorithm for optimizing the packet size used in wireless ARQ protocols. *Wireless Networks 5*, 1999.

[8] D. Zhang and K. M. Wasserman. Energy efficient data communication over fading channels. *Proc. IEEE WCNC*, 2000.

[9] W. Feller. *An Introduction to Probability Theory and Its Applications, Volume I.* John Wiley & Sons, 1968.

[10] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5, 1996.

[11] P. Erdös and P. Révész. On the Length of the Longest Head-Run. *Colloq. Math. Soc. J. Bolyai, Topics in Information Theory, Keszthely (Hungary)*, 1975.

[12] L. Gordon, M. F. Schilling, and M. S. Waterman. An Extreme Value Theory for Long Head Runs. *Probab. Th. Rel. Fields, Springer Verlag*, 72:279–287, 1986.

[13] R. Courant and F. John. *Introduction to Calculus and Analysis, Volume I.* Springer, 1989.